Supersymmtric Quantum Mechanics and Lefschetz fixed-point formula

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Abstract

We review the explicit derivation of the Gauss-Bonet and Hirzebruch formulae by physical model and give a physical proof of the Lefschetz fixed-point formula by twisting boundary conditions for the path integral.

1 Introduction

Physical models with supersymmetry are known to have close relation with mathematical structures. The Atiyah-Singer formula[1] for the index of Dirac operators as well as the index formulae for Euler number, signature, Todd genus, Hirzebruch χ_y-genus can be derived from a simple supersymmetric quantum mechanical model [3, 4, 5].

In this paper we show how to explicitly derive the Gauss-Bonet , Hirzebruch and Lefschetz fixed-point formulae from the supersymmetric sigma model and point out that the choices of boundary conditions in the path are crucial to the path integral calculation and can be realized as classical operators on the geometric objects. A special and unfamiliar boundary condition is chosen in order to derive the Lefschetz fixed-point formula.

The organization of the paper is as follows: In Sect.2, we give a quick review of supersymmetric sigma model on Riemannian manifold that we will use throughout this paper. In Sect.3 we use this model to derive Gauss-Bonet, Hirzebruch and Leftschetz fixed-point formulae.

2 Supersymmetric Quantum Mechanics on Riemannian manifold

We give a short review of Witten's definition of Supersymmetric Quantum Mechanics (SQM) in the Appendix. In this section, we study a simple but important example of SQM defined on a Riemannian manifold. For more details consult [6].

Let (M, g) be an oriented and compact Riemannian Manifold of dimention n with metric g. $I = [0, T]$ is the time interval. The bosonic field defines a map

$$\phi : I \longrightarrow M, \tag{2.1}$$

which is represented locally as $x^I \circ \phi = \phi^I$. Here $\{x^I\}, I = 1, \ldots, n$, is the local coordinate on M. The fermionic variables define sections

$$\psi, \bar{\psi} \in \Gamma(I, \phi^* TM \otimes C), \tag{2.2}$$

We can construct a Lagrangian for the bosonic and fermionic fields

$$L = \frac{1}{2} g_{IJ} \dot{\phi}^I \dot{\phi}^J + \frac{i}{2} g_{IJ}(\phi) \left(\bar{\psi}^I D_t \psi^J - D_t \bar{\psi}^I \psi^J \right) - \frac{1}{4} R_{IJKL} \psi^I \psi^J \bar{\psi}^K \bar{\psi}^L \tag{2.3}$$

where $D_t \psi^I = \frac{\partial}{\partial t} \psi^I + \Gamma^I_{JK} \dot{\phi}^J \psi^K$ and the summation convention is used. This lagrangian preserves the SUSY transformation

$$\delta \phi^I = \epsilon \bar{\psi}^I - \bar{\epsilon} \psi^I \tag{2.4}$$

$$\delta \psi^I = \epsilon \left(i \dot{\phi}^I - \Gamma^I_{JK} \bar{\psi}^J \psi^K \right) \tag{2.5}$$

$$\delta \bar{\psi}^I = \bar{\epsilon} \left(-i \dot{\phi}^I - \Gamma^I_{JK} \bar{\psi}^J \psi^K \right) \tag{2.6}$$

The supersymmetry charges are given by

$$Q = i g_{IJ} \bar{\psi}^I \dot{\phi}^J = i \bar{\psi}^I P_I \tag{2.7}$$

$$\bar{Q} = -i g_{IJ} \psi^I \dot{\phi}^J = -i \psi^I P_I \tag{2.8}$$

where $P_I = g_{IJ} \dot{\phi}^J$ is the conjugate momentum of ϕ^I.

After the canonical quantization ,we will have

$$\left[\phi^I, P_J \right] = i \delta^I_J \tag{2.9}$$

$$\left\{ \psi^I, \bar{\psi}^J \right\} = g^{IJ} \tag{2.10}$$

the Fermion number operator is

$$F = g_{IJ} \bar{\psi}^I \psi^J \tag{2.11}$$

The natural Hilbert space equipped with this quantum mechanical system is

$$F = \Omega^*(M) \otimes C \tag{2.12}$$

with the Hermitian inner product

$$(\omega_1, \omega_2) = \int_M \bar{\omega}_1 \wedge *\omega_2 \tag{2.13}$$

The obserbables are represented on this Hilbert space as familiar operators

$$\begin{aligned} \phi^I &= x^I \times \\ P_I &= -i\nabla_I \\ \bar{\psi}^I &= dx^I \wedge \\ \psi^I &= g^{IJ} i_{\frac{\partial}{\partial x^J}} \end{aligned}$$

Denote the vacuum $|0>$ satisfying $\psi^I|0>=0, \forall I$, then

$$\begin{array}{rcl} |0> & \leftrightarrow & 1 \\ \bar{\psi}^I|0> & \leftrightarrow & dx^I \\ & \vdots & \\ \bar{\psi}^1 \cdots \bar{\psi}^n|0> & \leftrightarrow & dx^1 \wedge \cdots dx^n \\ Q = i\bar{\psi}^I P_I & \leftrightarrow & d = dx^I \wedge \nabla_I \\ Q^\dagger = -i\psi^I P_I & \leftrightarrow & d^\dagger = -g^{IJ} i_{\frac{\partial}{\partial x^J}} \wedge \dfrac{\partial}{\partial x^I} \\ H = \dfrac{1}{2}\left\{Q, \bar{Q}\right\} & \leftrightarrow & \dfrac{1}{2}\triangle = \dfrac{1}{2}\left\{dd^\dagger + d^\dagger d\right\} \end{array}$$

The supersymmetric ground states are obviously given by

$$H_{(0)} = H(M,g) = \bigoplus_{p=0}^{n} H^p(M,g) \tag{2.14}$$

where $H(M,g)$ is the space of harmonic forms of the Riemannian manifold (M,g) and $H^p(M,g)$ is the space of harmonic p-forms.

On the other hand we know that the space of supersymmtric ground states is isomorphic to the Q-cohomology which here is represented by the De Rham cohomology. Take care of the grading by fermion number we will have

$$H^p(Q) = H^p_{DR}(M) \tag{2.15}$$

We get the famous formula

$$H^p(M,g) \cong H^p_{DR}(M) \tag{2.16}$$

The supersymmetric index is the Euler characteristic of the Q-complex, namely

$$Tr(-1)^F e^{-\beta H} = \sum_{p=0}^{n} (-1)^p dim H^p(Q) = \sum_{p=0}^{n} (-1)^p dim H^p_{DR}(Q) = \chi(M) \tag{2.17}$$

3 Mathematical application of SQM

3.1 Gauss-Bonnet foumula

We consider the SQM model on a Riemannian manifold (M, g) discussed in section 2. The lagrangian is

$$L = \frac{1}{2} g_{IJ} \dot{\phi}^I \dot{\phi}^J + i g_{IJ}(\phi) \bar{\psi}^I D_t \psi^J - \frac{1}{4} R_{IJKL} \psi^I \psi^J \bar{\psi}^K \bar{\psi}^L \tag{3.1}$$

Note that we choose a convenient lagrangian here which differs from eqs.(2.3) by a total derivative. After wick rotation $t \to -it$, we get

$$S_E = \int_0^\beta dt \, \left\{ \frac{1}{2} g_{IJ} \dot{\phi}^I \dot{\phi}^J + g_{IJ} \bar{\psi}^I D_t \psi^J + \frac{1}{4} R_{IJKL} \psi^I \psi^J \bar{\psi}^K \bar{\psi}^L \right\} \tag{3.2}$$

Since the Witten index is independent of β, we take $\beta \to 0$ and eqs.(3.2) shows that the path integral will localize on the constant maps. We can fourier expand the fields near the constant maps

$$\dot{\phi}^I = x_0^I + \sqrt{\beta} \sum_{n \neq 0}^{\infty} a_n^I e^{2\pi n t i/\beta} \tag{3.3}$$

$$\bar{\psi}^I = \beta^{1/4} \bar{\psi}_0^I + \sum_{n \neq 0}^{\infty} \bar{\psi}_0^I e^{2\pi n t i/\beta} \tag{3.4}$$

$$\psi^I = \beta^{1/4} \psi_0^I + \sum_{n \neq 0}^{\infty} \psi_0^I e^{2\pi n t i/\beta} \tag{3.5}$$

The factors of $\sqrt{\beta}$ and $\beta^{1/4}$ in the mode expansion is included to keep the first order approximation and remove the β-dependence from the integration measure. We will integrate out first all the nonzero modes of all fields and then all zero modes. Since the path integral is invariant under change of variables, we can use normal coordinates centered at the point where we do the expansion. Take the above expansion back to eqs.(3.2), we get

$$S_E = \sum_{n \neq 0} \left\{ \frac{(2\pi n)^2}{2} a_n^I (a_n^I)^* + (2\pi n i) \bar{\psi}_n^I \psi_n^I \right\} + \frac{1}{4} R_{IJKL}(x_0^I) \psi_0^I \psi_0^J \bar{\psi}_0^K \bar{\psi}_0^L + O(\beta) \tag{3.6}$$

The integration of nonezero modes will give 1 which is in fact a consequence of supersymmetry. The integration of zero modes gives

$$\chi(M) = Tr(-1)^F e^{-\beta H} = \frac{1}{(2\pi)^{n/2}} \int d(Vol) \int \prod_m d\psi_0^m d\bar{\psi}_0^m exp \left\{ -\frac{1}{4} R_{IJKL}(x_0^I) \psi_0^I \psi_0^J \bar{\psi}_0^K \bar{\psi}_0^L \right\}$$

where n is the dimension of the manifold. If n is odd, the above equation gives zero; if n is even ,the expression becomes

$$\begin{aligned} \chi(M) &= Tr(-1)^F e^{-\beta H} \\ &= \frac{(-1)^{n/2}}{2^n \left(\frac{n}{2}\right)! \pi^{n/2}} \int d(Vol) \varepsilon^{I_1 J_1 \dots I_m J_m} \varepsilon^{K_1 L_1 \dots K_m L_m} R_{I_1 J_1 K_1 L_1} \cdots R_{I_m J_m K_m L_m}, \ n = 2m, \end{aligned}$$

which is just the Gauss-Bonet formula [7].

3.2 Hirzebruch signature

A path integral derivation of χ_y-genus is given by [5]. We use the similar procedure as [5] to give an explicit calculation of Hirzebruch signature. Consider SQM on a Riemannian manifold (M, g)

$$L = \frac{1}{2} g_{IJ} \dot{\phi}^I \dot{\phi}^J + i g_{IJ}(\phi) \bar{\psi}^I D_t \psi^J - \frac{1}{4} R_{IJKL} \psi^I \psi^J \bar{\psi}^K \bar{\psi}^L \tag{3.7}$$

Notice that the lagrangian has a discrete symmetry $\bar{\psi} \leftrightarrow \psi$. Let the operator Γ implementing this symmetry. Recall that the hodge star operator acts on differential forms as

$$* : \Omega^p M \longrightarrow \Omega^{n-p} M \tag{3.8}$$

for $\varphi = \frac{1}{p\,!} \sum\limits_{i_1,\ldots,i_p} \varphi_{i_1,\ldots,i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$,

$$*\varphi = \frac{1}{(n-p)!} \sum_{i_1,\ldots,i_n} \frac{1}{p\,!} \varepsilon_{i_1,\ldots,i_n} \varphi^{i_1,\ldots,i_p} dx^{i_{p+1}} \wedge \cdots \wedge dx^{i_n} \tag{3.9}$$

where

$$\varphi^{i_1,\ldots,i_p} = \sum_{j1,\ldots,j_p} g^{i_1 j_1} \cdots g^{i_p j_p} \varphi_{j_1,\ldots,j_p} \tag{3.10}$$

In particular , $*1 = d(Vol)$, where $d(Vol)$ is the volume form

$$\begin{aligned} d(Vol) &= \sqrt{det(g)} dx^1 \wedge \cdots \wedge dx^n \\ &= \frac{1}{n!} \varepsilon_{i_1,\ldots,i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n} \end{aligned}$$

The first obserbation is that the operator Γ we defined exchanges ψ and $\bar{\psi}$ and therefore sends the vacuum state $|0>$ to the state which is annihilated by $\bar{\psi}$. Represented on the differential forms, this state is nothing but the volume form $d(Vol)$, so we have

$$\Gamma|0> = \sqrt{det(g)} \bar{\psi}^1 \cdots \bar{\psi}^n |0> \tag{3.11}$$

for the general state

$$\varphi = \frac{1}{p\,!} \varphi_{i_1,\ldots,i_p} \bar{\psi}^{i_1} \cdots \bar{\psi}^{i_p} |0> \tag{3.12}$$

the action by Γ gives

$$\begin{aligned} \Gamma\varphi &= \frac{1}{p\,!} \varphi_{i_1,\ldots,i_p} \psi^{i_p,\ldots,i_1} d(Vol) && (3.13) \\ &= \frac{1}{p\,!} \varphi_{i_1,\ldots,i_p} \psi^{i_p,\ldots,i_1} \frac{1}{n!} \varepsilon_{j_1,\ldots,j_n} \bar{\psi}^{j_1} \cdots \bar{\psi}^{j_n} |0> && (3.14) \\ &= \frac{1}{p\,!} \varphi_{i_1,\ldots,i_p} \frac{1}{(n-p)!} g^{i_1 j_1} \cdots g^{i_p j_p} \varepsilon_{j_1,\ldots,j_n} \bar{\psi}^{j_{p+1}} \cdots \bar{\psi}^{j_n} |0> && (3.15) \\ &= *\varphi && (3.16) \end{aligned}$$

So if represented on the differential forms , the operator Γ is just the hodge star operator. As a consequence, the Hirzebruch signature can be expressed by

$$\tau(M) = Tr\Gamma(-1)^F e^{-\beta H} = n^{E=0}(\Gamma = +1) - n^{E=0}(\Gamma = -1) \tag{3.17}$$

We come to the path integral calculation of Hirzebruch signature. To implement the discrete symmetry, we redefine the fermionic field

$$\psi_+^I = \frac{1}{2}\left(\bar{\psi}^I + \psi^I\right) \tag{3.18}$$

$$\psi_-^I = \frac{1}{2}\left(\bar{\psi}^I - \psi^I\right) \tag{3.19}$$

and the lagrangian takes the form

$$L = \frac{1}{2}g_{IJ}\dot{\phi}^I\dot{\phi}^J + ig_{IJ}\psi_+^I D_t\psi_+^J - ig_{IJ}\psi_-^I D_t\psi_-^J - \frac{1}{4}R_{IJKL}\psi_+^I\psi_+^J\psi_-^K\psi_-^L \tag{3.20}$$

the discrete symmetry is transformed to

$$\psi_+ \longleftrightarrow \psi_+ \tag{3.21}$$

$$\psi_- \longleftrightarrow -\psi_- \tag{3.22}$$

Now by a standard argument ,

$$Tr\Gamma(-1)^F e^{-\beta H} = \int_{BC} D\phi D\psi_+ D\psi_- e^{-S_E} \tag{3.23}$$

the boundary condition is just

$$\phi(\beta) = \phi(0) \tag{3.24}$$

$$\psi_+(\beta) = \psi_+(0) \tag{3.25}$$

$$\psi_-(\beta) = -\psi_-(0) \tag{3.26}$$

After wick rotation $t \to -it$,

$$S_E = \int_0^\beta dt \frac{1}{2}g_{IJ}\dot{\phi}^I\dot{\phi}^J + g_{IJ}\psi_+^I D_t\psi_+^J - g_{IJ}\psi_-^I D_t\psi_-^J + \frac{1}{4}R_{IJKL}\psi_+^I\psi_+^J\psi_-^K\psi_-^L \tag{3.27}$$

A subtle fact is that ψ_- has no constant mode by the boundary conditon (3.26) so we should take care of the first order to do the approximation. As before we do the fourier expansion

$$\phi^I = x_0^I + \sqrt{\beta}\sum_{n\neq 0}^{\infty} a_n^I e^{2\pi nit/\beta} \tag{3.28}$$

$$\psi_+^I = \sqrt{\frac{i}{2\pi\beta}}\psi_0^I + \sum_{n\neq 0}^{\infty}\psi_n^I e^{2\pi nit/\beta} \tag{3.29}$$

$$\psi_-^I = \frac{1}{2}e^{-\frac{i}{2}\pi + \pi ti}\eta_0^I + \sum_{n\neq 0}^{\infty}\eta_n^I e^{2\pi nit/\beta} \tag{3.30}$$

Choose the normal coordinate near x_0^I , substitute this back to eqs.(3.27) , take care of the connection term in D_t, keep the lowest order, we get

$$\begin{aligned} S_E &= \frac{1}{2}\sum_{n\neq 0}(2\pi n)^2 a_n^I (a_n^I)^* + \sum_{n\neq 0}(2\pi n i)\psi_{-n}^I\psi_n^I - \sum_{n\neq 0} 2n\pi i \eta_{-n-1}^I \eta_n^I \\ & -\sum_{n\neq 0}\frac{n}{2}\Omega_{KL}(a_n^K)^* a_n^L + \sum_{n\neq 0}\left(\frac{i}{2\pi}\Omega_{KL} - \pi i \delta_{KL}\right)\eta_{-n-1}^K\eta_n^L \\ & + e^{\frac{i}{2}\pi}\left(-\frac{i}{4\pi}\Omega_{KL} + \frac{\pi i}{2}\delta_{KL}\right)\eta_{-1}^K\eta_0^L + O(\beta) \end{aligned}$$

where

$$\Omega_{KL} = R_{IJKL}\psi_0^I\psi_0^J \tag{3.31}$$

Integrate out $a_n, (a_n)^\dagger$ gives

$$\prod_{n>0} det^{-1}\left((2\pi n)^2 I - n\Omega\right) det^{-1}\left((2\pi n)^2 I + n\Omega\right) \tag{3.32}$$

$$\tag{3.33}$$

Integrate out ψ_n, η_n gives

$$\begin{aligned} &\left\{\prod_{n>0} det\left((2\pi n)^2 I\right)\right\} det\left(e^{\frac{i}{2}\pi}\left(\frac{i}{4\pi}\Omega - \frac{\pi i}{2}I\right)\right) \\ &\left\{\prod_{n>0} det\left(2\pi n i - \left(\frac{i}{2\pi}\Omega - \pi i I\right)\right) det\left(-2\pi n i - \left(\frac{i}{2\pi}\Omega - \pi i I\right)\right)\right\} \end{aligned}$$

Put it together , we have

$$det\left(e^{\frac{i}{2}\pi}\left(\frac{i}{4\pi}\Omega - \frac{\pi i}{2}I\right)\right)\prod_{n>0} det^{-1}\left(I + \left(\frac{i\Omega/4\pi}{n\pi}\right)^2\right) det\left(I + \left(\frac{i}{4\pi}\Omega - \frac{\pi i}{2}I\right)^2\right)$$

To evaluate the above expression , we assume that $\frac{i}{4\pi}\Omega$ has eigenvalues χ_I and use the formula

$$\sinh x = x\prod_{n>0}\left(1 + \left(\frac{x}{n\pi}\right)^2\right) \tag{3.34}$$

$$\cosh x = e^{i\pi/2}\left(x - \frac{i}{2}\pi\right)\prod_{n>0}\left(1 + \left(\frac{(x - \frac{i}{2}\pi)}{n\pi}\right)^2\right) \tag{3.35}$$

we get

$$\prod_I \frac{\chi_I}{\tanh\chi_I} \tag{3.36}$$

Finally we integrate ψ_0^I and x_0^I and get the result for path integral calculation

$$\tau(M) = Tr\Gamma(-1)^F e^{-\beta H} = \int_M d(Vol) \int (d\psi_0^I) \prod_I \frac{\chi_I}{\tanh \chi_I} \tag{3.37}$$

which is just Hirzebruch's formula for the signature of the manifold.

3.3 Lefschetz fixed-point theorem

In this section , we use the path integral method to prove the Lefschetz fixed-point formula. Let $f : M \to M$ be a smooth map of a compact oriented manifold into itself. Denote by $H^q(f)$ the induced map on the cohomology $H^q(M)$. The *Lefschetz number* of f is defined to be

$$L(f) = \sum_q (-1)^q Tr H^q(f) \tag{3.38}$$

At a fixed point P of f the derivative $(Df)_P$ is an endomorphism of the tangent space $T_P M$. The *multiplicity* of the fixed P is defined to be

$$\sigma_P = \text{sgn } det((Df)_P - I) \tag{3.39}$$

Recall that the Lefschetz fixed-point formula states that

$$L(f) = \sum_P \sigma_P \tag{3.40}$$

If f is generated by a vector field on M , (3.40) can be proved by modifying the action of SQM on M by a vector field term while still preserving the supersymmtry [3]. We show that eqs.(3.40) can also be derived directly from the standard SQM on Riemannian manifold.

We start again with the lagrangian

$$L = \frac{1}{2} g_{IJ} \dot{\phi}^I \dot{\phi}^J + \frac{i}{2} g_{IJ}(\phi) \bar{\psi}^I D_t \psi^J - \frac{i}{2} g_{IJ} D_t(\phi) \bar{\psi}^I \psi^J - \frac{1}{4} R_{IJKL} \psi^I \psi^J \bar{\psi}^K \bar{\psi}^L \tag{3.41}$$

We first have a close look at how $H^q(f)$ acts on the state

$$\begin{aligned} & H^q(f) \sum_{i_1,\ldots,i_p} \varphi_{i_1,\ldots,i_p}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\ = & \sum_{i_1,\ldots,i_p} \varphi_{i_1,\ldots,i_p}(f(x)) df^{i_1}(x) \wedge \cdots \wedge df^{i_1}(x) \end{aligned}$$

which just has the effect of sending

$$\phi \longrightarrow f(\phi) \tag{3.42}$$
$$\bar{\psi} \longrightarrow f_*(\bar{\psi}) \tag{3.43}$$

Recall that $\bar{\psi}$ is regarded as a smooth section of TM.

So by the standard argument in quantum field theory , we have a path integral expression for $L(f)$

$$L(f) = TrH^*(f)(-1)^F e^{-\beta H} \tag{3.44}$$
$$= \int_{BC} D\phi D\bar{\psi} D\psi e^{-S_E} \tag{3.45}$$

where the boundary condition is

$$\phi(\beta) = f(\phi(0)) \tag{3.46}$$
$$\bar{\psi}(\beta) = f_*(\bar{\psi}(0)) \tag{3.47}$$
$$\psi(\beta) = \psi(0) \tag{3.48}$$

the Euclidean action is the same as before

$$S_E = \int_0^\beta dt \left\{ \frac{1}{2} g_{IJ} \dot{\phi}^I \dot{\phi}^J + \frac{1}{2} g_{IJ} \bar{\psi}^I D_t \psi^J - \frac{1}{2} g_{IJ} D_t \bar{\psi}^I \psi^J + \frac{1}{4} R_{IJKL} \psi^I \psi^J \bar{\psi}^K \bar{\psi}^L \right\}$$

The path integral still localizes to the constant maps which is just the constant map to fixed point of f due to the boundary conditions (3.46) . We do the mode expansion respecting the boundary condition

$$\phi^I = \sqrt{\beta}(tf^I(x_0) + (1-t)x_0^I) + \sqrt{\beta} \sum_{n \neq 0} a_n^I e^{2\pi nit/\beta} \tag{3.49}$$

$$\bar{\psi}^I = \sqrt{2} \left(e^{tA}\right)^I_J \bar{\psi}_0^J + \sum_{n \neq 0}^{\infty} \bar{\psi}_n^I e^{2\pi nit/\beta} \tag{3.50}$$

$$\psi^I = \sqrt{2} \psi_0^I + \sum_{n \neq 0}^{\infty} \psi_n^I e^{2\pi nit/\beta} \tag{3.51}$$

where formally the matrix A is related to f by

$$e^A = Df(x_0) \tag{3.52}$$

Choose the normal coordinate and take the expansion back to the action, keeping only the first order, we get

$$S_E = \frac{1}{2} x_0^I (\partial_I f(x_0)^K - \delta_I^K)(\partial_K f(x_0)^J - \delta_K^J) x_0^J - \bar{\psi}_0^J \left(\left(e^A\right)^I_J - \delta_J^I \right) \psi_0^I$$
$$+\text{excited modes} + \text{higer terms}$$

The excited modes of boson and fermion will cancel each other and we left with the zero modes . The integration of x_0 near the fixed point of f gives

$$\sqrt{det^{-1}\left((Df-I)^t (Df-I)\right)} \tag{3.53}$$

the integration of fermion zero modes gives

$$det\,(Df - I) \tag{3.54}$$

All together we have

$$\begin{aligned} L(f) &= \sum_q (-1)^q TrH^q(f) \\ &= TrH^*(f)(-1)^F e^{-\beta H} \\ &= \int_{BC} D\phi D\bar{\psi} D\psi e^{-S_E} \\ &= \sum_P \frac{det\,(Df|_P - I)}{\sqrt{det\,((Df|_P - I)^t (Df|_P - I))}} \\ &= \sum_P \sigma_P \end{aligned}$$

A Supersymmetric Quantum Mechanics

A.1 Witten's definition

Consider a quantum mechanical system consisting of a Hilbert (Fock) space F and Hamiltonian H. The system is said to be supersymmetric quantum mechanical (SQM) if

1.F has a decomposition $F = F^B \oplus F^F$ and states in F^B and F^F are called bosonic and fermionic states respectively. There is an operator $(-1)^F$ such that

$$(-1)^F \Psi = \Psi \quad if \ \Psi \in F^B \tag{A.1}$$

$$(-1)^F \Psi = -\Psi \quad if \ \Psi \in F^F \tag{A.2}$$

F and $(-1)^F$ are called fermion number operator and chirality operator.

2.There are N operators Q^I, I=1,$\cdots$,N, such that

$$Q^I, Q^{I\dagger} \quad : \quad F^B \to F^F, \tag{A.3}$$

$$Q^I, Q^{I\dagger} \quad : \quad F^F \to F^B, \tag{A.4}$$

$$\left\{(-1^) F, Q^I\right\} = \left\{(-1^) F, Q^{I\dagger}\right\} = 0 \tag{A.5}$$

Q^I are called supersymmetry(SUSY) charges or generators.

3.The SUSY generators satisfy the general superalgebra condition:

$$\left\{Q^I, Q^{J\dagger}\right\} = 2\delta^{IJ} H \tag{A.6}$$

$$\left\{Q^I, Q^J\right\} = \left\{Q^I, Q^J\right\} = 0 \tag{A.7}$$

where I,J=1,$\cdots$,N.
A quantum system satisfying the above conditions is said to have a type N supersymmetry.

A.2 The general structure of Supersymmetric Quantum Mechanics

To be simple, we assume that the SQM we consider preserve one supercharge Q. The result for the general case of arbitrary number of supercharges is similar.

Property A.1. *The Hamiltonian is a non-negative operator*

Proof:

$$H = \frac{1}{2}\left\{Q, Q^\dagger\right\} \geq 0 \tag{A.8}$$

$\square$

Property A.2. *A state has zero energy if and only if it is annihilated by Q and $Q^\dagger$:*

$$H\Psi = 0 \Longleftrightarrow Q\Psi = Q^\dagger\Psi = 0 \textit{ for } \Psi \in F \tag{A.9}$$

Proof:
if we have

$$\begin{aligned} &H\Psi = 0 \text{ for } \psi \in F \\ &\Longrightarrow (\Psi, H\Psi) = 0 \end{aligned}$$

by eqs.(A.8) we have

$$\begin{aligned} & & (\Psi, H\Psi) \\ & = & \left(\Psi, \left\{Q, Q^\dagger\right\}\Psi\right) \\ & = & (Q\Psi, Q\Psi) + \left(Q^\dagger\Psi, Q^\dagger\Psi\right) = 0 \\ \Longrightarrow & & \\ & & Q\Psi = Q^\dagger\Psi = 0 \end{aligned}$$

by the positivity of the inner product on the Hilbert space F. The converse case is trivial due to eqs.(A.8). $\square$

In the physical language, this property is restated as: *The zero energy ground state is a supersymmetric state and vice versa.* Thus we also call such a state a *supersymmetric ground state.*

The Hilbert space can be decomposed in terms of eigenspaces of the Hamiltonian

$$F = \bigoplus_{n=0,1,\ldots} F_{(n)}, \quad H\mid_{F_{(n)}} = E_n. \tag{A.10}$$

We accept the convention that $E_0 = 0 < E_1 < E_2 < \cdots$. Since Q, $Q^\dagger$ and $(-1)^F$ commute with the Hamiltonian, these operators preserve the energy levels:

$$Q, Q^\dagger, (-1)^F : F_{(n)} \longrightarrow F_{(n)} \tag{A.11}$$

In particular, each energy level $F_{(n)}$ is decomposed into even and odd (or bosonic and fermionic) subspaces

$$F_{(n)} = F^B_{(n)} \oplus F^F_{(n)}, \tag{A.12}$$

and the supercharges map one subspace to the other:

$$Q, Q^\dagger : F^B_{(n)} \longrightarrow F^F_{(n)}, F^F_{(n)} \longrightarrow F^B_{(n)} \tag{A.13}$$

Consider the combination $Q_1 := Q + Q^\dagger$,which obeys

$$Q_1^2 = 2H \tag{A.14}$$

This operator preserves each energy level, mapping $F^B_{(n)}$ to $F^F_{(n)}$ and vice versa. Since $Q_1^2 = 2E_n$ at the nth level, as long as $E_n > 0$, Q_1 is invertible and we get the following

Property A.3.

$$F^B_{(n)} \cong F^F_{(n)} \quad \textit{for } n > 0 \tag{A.15}$$

Note that the bosonic and fermionic supersymmetric ground states do not have to be paired since at the zero energy level $F_{(0)}$, the operator Q_1 squares to zero. *Witten index* is defined to be

$$Tr(-1)^F e^{-\beta H} = dim F^B_{(0)} - dim F^F_{(0)} \tag{A.16}$$

which is independent of β by eqs.(A.15). Physically, the importance of witten index is that it is invariant under continuous deformation of the theory(because the states move in pairs due to the isomorphism eqs.(A.15)) and give a condition for the broken of supersymmetry.

Now we come to the mathematical structure. Since $Q^2 = 0$, we have a Z_2-graded complex of vector spaces

$$F^F \xrightarrow{Q} F^B \xrightarrow{Q} F^F \xrightarrow{Q} F^B \tag{A.17}$$

and thus we can consider the cohomology of this complex,

$$H^B(Q) := \frac{KerQ : F^B \to F^F}{ImQ : F^F \to F^B} \tag{A.18}$$

$$H^F(Q) := \frac{KerQ : F^F \to F^B}{ImQ : F^B \to F^F} \tag{A.19}$$

The complex in eqs.(A.17) also decomposes into energy levels. At the excited level $F_{(n)}$, we have

$$\left(QQ^{\dagger}+Q^{\dagger}Q\right)/(2E_n)=1 \tag{A.20}$$

which implies that the Q-cohomology at the excited level is trivial. However,at the zero energy level $F_{(0)}$, the coboundary operator is trivial, $Q=0$, and we get

Property A.4.

$$H^B(Q)=F^B_{(0)},H^F(Q)=F^F_{(0)} \tag{A.21}$$

Finally, we provide the path-integral expression for the Witten index. By a general physical argument, we have

Property A.5.

$$Tr(-1)^F=Tr(-1)^Fe^{-\beta H}=\int_P DXD\psi D\psi e^{-S_E(X,\psi,\bar{\psi})} \tag{A.22}$$

where X denotes the bosonic field and $\psi,\bar{\psi}$ denote the fermionic field. P on the measure means that we impose the periodic boundary conditions:

$$X(\beta)=X(0),\psi(\beta)=\psi(0),\bar{\psi}(\beta)=\bar{\psi}(\beta) \tag{A.23}$$

References

[1] M.F.Atiyah and I.M.Singer, Bull.Am.Math.Soc.69 (1963), 422; M.F.Atiyah and I.M.Singer, Ann, Math.87(1968), 484; ibid., 546.

[2] E.Witten:*Supersymmetry and morse theory*. J.Diff.Geom.17,661(1982).

[3] L.Alvarez-Gaume,*Supersymmetry and the Atiyah-Singer index theorem*,Comm.Math.Phys.90(1983),161.

[4] D.Friedan and P.Windey,*Supersymmetric derivation of the Atiyah-Singer index theorem and the chiral anomaly*,Nucl.Phys.B235(1984),395.

[5] M.Guowu ,*A path integral derivation of χ_y-genus*, J. Phys. A: Math. Gen. 36(2003) 1083-1086,math-ph/0306041.

[6] K.Hori, S.Hatz, A.Klemm, R.Pandharipande, R.Thomas, C.Vafa, R.Vakil and E.Zaslow, "Mirrow Symmetry". Clay Mathematics Monographs 1, American Mathematical Societry, Providence, Clay Mathematics Institute, Cambridge, 2003, ISBN 0-8218-2955-6.

[7] S.S.Chern , On the curvature integrals in a Riemannian manifold. Ann. Math. 46 , 674(1942)